Some problems about the uncountable Specker phenomenon (Interplay between large cardinals and small cardinals)

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1 Introduction

We are motivated from [3]. We find a mistake in [3] and improve it partially. The uncountable Specker phenomenon in the commutative case was studied around 1955. J. Łoś and E.C.Zeeman [8] independently showed that $\mathbb{Z}^{\kappa}$ exhibits the Specker phenomenon if κ is less than the least measurable cardinal. There is a similar result in the non-commutative case. S.Shelah and K.Eda [6] showed that the unrestricted free product $\varprojlim(*_{i\in X}G_{i}, p_{XY}$: $X\subseteq Y\Subset I)$ exhibits the Specker phenomenon if the cardinality of the index set I is less than the least measurable cardinal. Some problems occur from the non-commutative case and we investigate them.

2 Definitions and Basics

Definition 2.1. Let G_{i} $(i\in I)$ be groups s.t $G_{i}\cap G_{j}=\{e\}$ for any $i\neq j\in I$. we call elements of $\bigcup_{i\in I}G_{i}\setminus\{e\}$ letters. A word W is a function

$W:\overline{W}\rightarrow\bigcup_{i\in I}G_{i}\backslash\{e\}$ $\overline{W}$ is a linearly ordered set and $\{\alpha\in\overline{W}\mid W(\alpha)\in G_{i}\}$ is finite for any $i\in I$. The class of all words is denoted by $\mathcal{W}(G_{i}:i\in I)$ (abbreviated by $\mathcal{W}$). In case the cardinality of $\overline{W}$ is countable, we say that W is a σ-word.

Definition 2.2. U and V are isomorphic, which is denoted by $U\equiv V$, if there exists an order isomorphism $\varphi:\overline{U}\rightarrow\overline{V}$ s.t $\forall\alpha\in\overline{U}$ $(U(\alpha)=V(\varphi(\alpha))$. It is easily seen that $\mathcal{W}$ becomes a set under this identification.

Definition 2.3. For a subset $X\subseteq I$, the restricted word W_{X} of W is given by the function

$W_{X}:\overline{W_{X}}\rightarrow\bigcup_{i\in I}G_{i}$ where $\overline{W_{X}}=\{\alpha\in\overline{W}|W(\alpha)\in\bigcup_{i\in X}G_{i}\}$ and $W_{X}(\alpha)=W(\alpha)$ for all $\alpha\in\overline{W_{X}}$. Hence $W_{X}\in \mathcal{W}$. If X is finite , then we can regard W_{X} as an element of the free product $*_{i\in X}G_{i}$.

Definition 2.4. U and V are equivalent, which is denoted by $U \sim V$, if $U_F = V_F$ for all $F \subset\subset I$ where we regard U_F and V_F as elements of the free product $*_{i\in F}G_i$.
So, "$U_F = V_F$" means that they are equal in the sense of the free product $*_{i\in F}G_i$.

Let $[W]$ be the equivalent class of a word W. The composition of two words and the inverse of a word are defined naturally. Thus $\mathcal{W}/\sim = \{[W] \mid W \in \mathcal{W}\}$ becomes a group.

Definition 2.5. $\times_{i\in I}G_i$ is the group $\mathcal{W}(G_i : i \in I)/\sim$. Clearly, if I is finite, then $\times_{i\in I}G_i$ is isomorphic to the free product $*_{i\in I}G_i$.

Definition 2.6. W is reduced if $W \equiv UXV$ implies $[X] \neq e$ for any non-empty word X where e is the identity, and for any contiguous elements α and β of $\overline{W}$, it never occurs that $W(\alpha)$ and $W(\beta)$ belong to the same G_i.

Definition 2.7. $l_i(W)$ is the cardinality of $\{\alpha \in X \mid X(\alpha) \in G_i\}$ where X is the reduced word of W.

Theorem 2.1. ([1] Theorem1.4.)For any word W, there exists a reduced word V such that $[W] = [V]$ and V is unique up to isomorphism.

Proposition 2.1. ([1] Proposition1.9.) If $g_\lambda(\lambda \in \Lambda)$ are elements of $\times_{i\in I}G_i$ such that $\{\lambda \in \Lambda \mid l_i(g_\lambda) \neq 0\}$ are finite for all $i \in I$, then there exists a natural homomorphism $\varphi : \times_{\lambda\in\Lambda}\mathbb{Z}_\lambda \to \times_{i\in I}G_i$ via $\delta_\lambda \mapsto g_\lambda$ $(\lambda \in \Lambda)$ where $\mathbb{Z}_\lambda(\lambda \in \Lambda)$ are copies of the integer group and δ_λ is the 1 of $\mathbb{Z}_\lambda$.

3 The non-commutative uncoutable Specker phenomenon

Theorem 3.1. *(S.Shelah and K.Eda [6]) Let S be a n-slender group. For any homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \to S$, there exist ω_1-complete ultrafilters $\mathcal{U}_1, \cdots, \mathcal{U}_n$ on I such that $h = \overline{h} \circ p_{u_1\cup\cdots\cup u_n}$ for any $u_1 \in \mathcal{U}_1, \cdots, u_n \in \mathcal{U}_n$. Moreover, if the cardinality of I is less than the least measurable cardinal, then h factors through some finitely generated free group.*

$$\begin{tikzcd}
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \arrow[r, "h"] \arrow[d, "p_{u_1\cup\cdots\cup u_n}"'] & S \\
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset u_1 \cup \cdots \cup u_n) \arrow[ur, dashed, "\exists\overline{h}"'] &
\end{tikzcd}$$

Let $\mathcal{F} = \{X|\exists u_1 \in \mathcal{U}_1 \cdots \exists u_n \in \mathcal{U}_n(\bigcup_{i\leq n} u_i \subseteq X)\}$. It becomes an ultrafilter on I. We introduce an equivalence relation $\sim_{\mathcal{F}}$ on $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)$. $x \sim_{\mathcal{F}} y$ if and only if there exists $u \in \mathcal{F}$ such that $p_u(x) = p_u(y)$. Then, we get the following diagram.

$$\begin{tikzcd}
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I) \arrow[r, "h"] \arrow[d] & S \\
\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{F} \arrow[ur, dashed, "\exists\overline{h}"'] &
\end{tikzcd}$$

It is a problem that what kind of group is $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{F}$. We remark that $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{F}$ could not be equal to $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{U}_1 * \cdots * \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset I)/\mathcal{U}_n$. For the first step, we consider the case $n = 1$ and we investigate its cardinality.

Definition 3.1. *Let $F, G \Subset \omega$ with $|F| = |G|$ and e_{FG} be the order isomorphism from F to G. Then, we naturally regard e_{FG} as an isomorphism from $*_{i\in F}\mathbb{Z}_i$ to $*_{i\in G}\mathbb{Z}_i$. An element $x \in \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega)$ is homogeneous if and only if for any $F, G \Subset \omega$ with $|F| = |G|$, $e_{FG}(p_F(x)) = p_G(x)$.*

Let H be the subgroup consisting of all homogeneous elements.

Theorem 3.2. *Let κ be a measurable cardinal and $\mathcal{U}$ be a κ-complete normal ultrafilter on κ. Then, $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U} \simeq H$.*

Proof. Let $\mathcal{U}^n = \{X \in [\kappa]^n|\exists u \in \mathcal{U}([u]^n \subseteq X)\}$ for $n \geq 2$ and $x \in \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)$. By the assumption, $\mathcal{U}^n$ is a κ-complete ultra filter for any n. Since $[\kappa]^n = \bigcup_{W\in *_{i<n}\mathbb{Z}_i}\{F|e_{Fn}(p_F(x)) = W\}$, there exist $W_{n,x} \in *_{i<n}\mathbb{Z}_i$ such that $\{F|e_{Fn}(p_F(x)) = W_{n,x}\} \in \mathcal{U}$. We define a homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U} \to H$ as $h([x])(n) = W_{n,x}$. It is easily seen that h is an isomorphism.

□

Proposition 3.1. *the cardinality of H is 2^ω. Moreover, H is not n-slender.*

To prove it, we need a lemma.

Definition 3.2. *Let $x_n \in \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega)$ for any $n < \omega$ and $[x_0, x_1] = x_0x_1x_0^{-1}x_1^{-1}$. We define inductively $[x_0, \cdots, x_n]$ as the following.*

$$[x_0, \cdots, x_{n+1}] := [x_0, \cdots, x_n]x_{n+1}[x_0, \cdots, x_n]^{-1}x_{n+1}^{-1}$$

Lemma 3.1. *There exists $y_n \in H(n < \omega)$ such that $\forall i < n(y_n(i) = e)$ for any $n < \omega$.*

Proof. Let δ_i be the 1 of $\mathbb{Z}_i$. We define y_n as the following.

$$\begin{aligned} y_n(n) &= [\delta_0, \cdots, \delta_{n-1}] \\ y_n(n+1) &= [\delta_0, \cdots, \delta_n][\delta_0, \cdots, \delta_{n-1}][\delta_0, \cdots, \delta_{n-2}, \delta_n] \cdots [\delta_1, \cdots, \delta_n] \\ \vdots &= \vdots \end{aligned}$$

We give a more precise definition. Let $A_{n+k,l} = \{f \in^l (n+k) | f \text{ is order preserving }\}$ with $n \le l \le n+k$. $A_{n+k,l} = \{f_i | i < l\}(i < j \to f_i < f_j)$ is linear ordered by the lexicographical order.

$$\prod_{f\in A_{n+k,l}} [\delta_{f(0)}, \cdots \delta_{f(l-1)}] := [\delta_{f_0(0)}, \cdots \delta_{f_0(l-1)}] \cdots [\delta_{f_{l-1}(0)}, \cdots \delta_{f_{l-1}(l-1)}]$$

$$y_n(n+k) = \prod_{f\in A_{n+k,n+k}}[\delta_{f(0)}, \cdots \delta_{f(l-1)}] \cdots \prod_{f\in A_{n+k,n}}[\delta_{f(0)}, \cdots \delta_{f(l-1)}]$$

Clearly, these are desired elements.

□

Proof of Propostion 3.1. Let $y_n(n < \omega)$ be as Lemma 3.1. There exists an homomorphism $h : \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega) \to \varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \omega)$ which maps δ_n to y_n for any $n < \omega$. Clearly, the image of h is contained by H. Therefore, H is not n-slender. By Lemma 2.6 in [4], we can conclude $|H| = 2^\omega$.

□

Proposition 3.2. *$H^* = \{W \in \times_{n<\omega}\mathbb{Z}_n | W$ is homogeneous$\}$ is n-slender.*

Proof. Firstly, we claim that $W \in H^* \setminus \{e\}$ implies $l_i(W) \neq 0$ for any $i < \omega$. Suppose the negation. Let n be the least natural number such that $W_{\{0,\cdots,n-1\}} \neq e$ and take $i < \omega$ with $l_i(W) = 0$. If $i < n$, then $W_{\{0,\cdots,n-1\}} = W_{n\setminus\{i\}}$. Since W is homogeneous, $e_{n\setminus\{i\}n-1}(W_{n\setminus\{i\}}) = W_{\{0,\cdots,n-2\}} \neq e$. It is a contradiction to the minimality of n. If $n \leq i$, we can deduce a contaradiction as well. Now, we show the n-slenderness of H^*. Assume not, then there exists a homomorphism $h : \times_{n<\omega}\mathbb{Z}_n \to H^*$ such that $h(\delta_n) \neq e$ for all $n < \omega$. By theorem 2.3 in [2], there exists a standard homomorphism $\overline{h}$ and $u \in \times_{n<\omega}\mathbb{Z}_n$ such that $h = u\overline{h}u^{-1}$. Because $\{n | l_0(\overline{h}(\delta_n)) \neq 0\}$ is finite, we can take N with $l_0(\overline{h}(\delta_N)) = 0$. On the other hand, $\overline{h}(\delta_N)$ is a non-trivial homogeneous word which is a contradiction. □

4 Problems

Question 4.1. *What is the cardinality of $\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U}$ when there are only finitely n such that $\mathcal{U}^n$ is an ultrafilter.*

In the proof of theorem 3.2, the fact $\mathcal{U}^n$ is a σ-complete ultrafilter for all n is essential. It is clear that $\mathcal{U}^{n+1}$ is an ultrafilter implies $\mathcal{U}^n$ is so. Therefore, the case there are only finitely n such that $\mathcal{U}^n$ is an ultrafilter is left. We conjecture $|\varprojlim(*_{i\in X}\mathbb{Z}_i, p_{XY} : X \subseteq Y \Subset \kappa)/\mathcal{U}| \geq \kappa$ in the case.

Question 4.2. *Is the cardinality of H^* countable or uncountable ?*

Unfortunately, we find that the proof of [3] about this problem is wrong and we can not improve it yet.

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